L^p ESTIMATES AND ASYMPTOTIC BEHAVIOR FOR FINITE ENERGY SOLUTIONS OF EXTREMALS TO HARDY-SOBOLEV INEQUALITIES

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Abstract. Motivated by the equation satisfied by the extremals of certain Hardy-Sobolev type inequalities, we show sharp L^q regularity for finite energy solutions of p-laplace equations involving critical exponents and possible singularity on a sub-space of $\mathbb{R}^n$, which imply asymptotic behavior of the solutions at infinity. In addition, we find the best constant and extremals in the case of the considered L^2 Hardy-Sobolev inequality.

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1. Introduction

This paper has three goals - prove sharp L^q regularity for solutions of nonlinear p-laplacian equations involving critical exponent and a singularity on a lower dimensional subspace, establish sharp rate of decay in the case $p = 2$, and finally determine the extremals in a related L^2 Hardy-Sobolev inequality. We indicate some other possible applications concerning stationary cylindrical states of the Vlasov-Poisson system and non-completeness of metrics with finite volume on some noncompact manifolds.

The organization of the paper is as follows. In Section 2 we shall study the L^p regularity and asymptotic behavior at infinity of non-linear equations involving critical growth. In its simplest form, our result is the following. For $1 < p < n$, we let $p' = \frac{p}{p-1}$ and $p^* = \frac{np}{n-p}$ be correspondingly the Hölder conjugate and the Sobolev conjugate exponents. Suppose $V \in L^{\frac{p^*}{p^*-p}}(\mathbb{R}^n)$, and if $p > 2$ assume further $V \geq 0$. Let u be a weak nonnegative solution in $\mathbb{R}^n$, i.e. u is of finite energy, cf. (2.6), of the inequality

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) \;\leq\; V u^{p-1}. \tag{1.1}$$

Then, $u \in L^q(\mathbb{R}^n)$ for any $\frac{p^*}{p'} < q \leq \infty$, which in general cannot be improved further, and for any $0 < \theta < 1$ there exists a positive constant C_θ, such that,

$$u(z) \;\leq\; \frac{C_\theta}{1 + |z|^{\theta\frac{n-p}{p-1}}}, \qquad z \in \mathbb{R}^n. \tag{1.2}$$

We shall actually consider inequalities with a possible singularity on a subspace of $\mathbb{R}^n$ in the right-hand side, which arise naturally in the study of the extremals of certain Hardy-Sobolev inequalities, see further below and (2.5) for the exact setting. Our results are contained in Theorems 2.1, 2.5, 2.8 and 2.9. The asymptotic behavior of solutions to elliptic equations with critical non-linearity has been studied extensively. Generally speaking, the results and the analysis depend on whether one assumes a priori that the solution has finite energy or one is given a smooth solution, see [S1], [S2], [SW], [E],[GS] and [Li], to name a few. We work under

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the assumption of finite energy. The paper extends the results of Egnell [E], which correspond to Theorems 2.1 and 2.8 without the possible singularity on a submanifold of $\mathbb{R}^n$.

In Sections 3 we consider the case $p = 2$. The main result here is that, under some natural additional assumption, u has the same asymptotic as the fundamental solution of the laplacian at infinity, see Theorem 3.1. The proof exploits the fact that $\mathbb{R}^n$ has a positive Yamabe invariant and thus the method will be applicable to cases when the ambient space does not have a conformal transformation with the properties of the Kelvin transform. Furthermore, we work in the setting in which V could have a singularity on a subspace, which renders results based on radial symmetry inapplicable.

Section 4 contains some calculations which show how one can construct explicit solutions of the Euler-Lagrange equation of certain Hardy-Sobolev inequalities. In the subsequent section we show that these solutions are extremals for which the best constant in the considered Hardy-Sobolev inequalities is achieved.

In Section 5, we determine the best constant in the Hardy-Sobolev embedding Theorem involving the distance to a subspace of $\mathbb{R}^n$. In order to explain the considered inequality we need a few notations, which shall be used throughout the paper. For $p \geq 1$ we define the space $\mathcal{D}^{1,p}(\mathbb{R}^n)$ as the closure of $C_o^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \; = \; \Big(\int_{\mathbb{R}^n} |\nabla u|^p dz\Big)^{1/p}. \tag{1.3}$$

Let $n \geq 3$ and $2 \leq k \leq n$. For a point z in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ we shall write $z = (x, y)$, where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$. The following Hardy - Sobolev inequality was proven in Theorem 2.1 of [BT].

Theorem 1.1 ([BT])**.** *Let $n \geq 3$, $2 \leq k \leq n$, and p, s be real numbers satisfying $1 < p < n$, $0 \leq s \leq p$, and $s < k$. There exists a positive constant $S_{p,s} = S(s, p, n, k)$ such that for all $u \in D^{1,p}(\mathbb{R}^n)$ we have*

$$\Big(\int_{\mathbb{R}^n} \frac{|u|^{\frac{p(n-s)}{n-p}}}{|x|^s}\, dz\Big)^{\frac{n-p}{p(n-s)}} \leq \; S_{p,s}\Big(\int_{\mathbb{R}^n} |\nabla u|^p\, dz\Big)^{\frac{1}{p}}. \tag{1.4}$$

When $k = n$ the above inequality becomes the Caffarelli-Kohn-Nirenberg inequality, see [CKN], for which the optimal constant $S_{p,s}$ was found in [GY]. The case $p = 2$ was considered earlier in [O] and [GMGT], where the inequality is written in an equivalent, but slightly different form. If we introduce the numbers $\sigma = \frac{1}{2}\frac{s(n-p)}{p(n-s)}$, hence $0 \leq \sigma < 1$, and $p_\sigma = \frac{n-p}{p(n-s)}$ the above inequality becomes

$$\Big(\int_{\mathbb{R}^n} \frac{|u|^{p_\sigma}}{|x|^{\sigma p_\sigma}}\, dz\Big)^{1/p_\sigma} \; \leq \; C\Big(\int_{\mathbb{R}^n} |\nabla u|^2\, dz\Big)^{1/2}, \tag{1.5}$$

where C is a positive constant. In the case $p = 2$ and $k = n$ the sharp constant was computed in [GMGT]. When $\sigma < 1$ the extremals were found by Lieb in [L], while when $\sigma = 1$ we have the classical Hardy inequality, which does not have extremal functions.

The main result of Section 5 is the proof of the following Theorem.

Theorem 1.2. *Suppose $n \geq 3$ and $2 \leq k \leq n$. There exists a positive constant $K = K_{n,k,2}$ such that for all $u \in D^{1,2}(\mathbb{R}^n)$ we have*

$$\Big(\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} \frac{|u|^{\frac{2(n-1)}{n-2}}}{|x|}\, dxdy\Big)^{\frac{n-2}{2(n-1)}} \; \leq \; K\Big(\int_{\mathbb{R}^n} |\nabla u|^2\, dz\Big)^{\frac{1}{2}}. \tag{1.6}$$

Furthermore, K is given in (5.8) *and the positive extremals are the functions*

$$v \;=\; \lambda^{-(n-2)} \Big(\frac{4}{(n-2)^2}\Big)^{-\frac{n-2}{2}} K^{-(n-1)} \Big((|x| + \frac{n-2}{4a\lambda^2})^2 + |y-y_o|^2 \Big)^{-\frac{n-2}{2}}, \tag{1.7}$$

where $\lambda \;>\; 0$, $y_o \in \mathbb{R}^{n-k}$.

Let us note that the results of Section 2 can be applied to the non-negative extremals of the general Hardy-Sobolev inequality of Theorem 1.1.

Finally, Section 6 contains some further simple applications, which indicate the direction of some future investigations.

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2. Regularity and asymptotic of weak solutions

The goal of this section is to prove sharp L^q regularity for solutions of the considered equations, which would then lead to bounds on the rate of decay at infinity.

We start with two definitions. Let p and s be as in Theorem 1.1 and denote by $p^*(s)$ the Hardy-Sobolev conjugate

$$p^*(s) \;=\frac{p(n-s)}{n-p} \tag{2.1}$$

and by p' the Hölder conjugate $p' \;=\; \dfrac{p}{p-1}$. There is another exponent, which will play an important role. For any s as above we define the exponent $r = r(s)$ to be the Holder conjugate of the exponent $r' = r'(s)$ defined by

$$r' \;=\; \frac{p^*}{p^*(s)-p}, \quad \text{thus} \quad r = \frac{n}{n-p+s}. \tag{2.2}$$

Notice that $1 \le r$, $0 \le rs \le p$, and furthermore we have the identity

$$rp \;=\; p^*(rs). \tag{2.3}$$

However, in general rs could be bigger than k, and thus due to the restriction $s<k$ in the Hardy-Sobolev inequality we shall consider only the case

$$s(n-k) \;<\; k(n-p), \tag{2.4}$$

which implies $rs<k$.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^n$, which is not necessarily bounded,* $1<p<n$, $0 \le s \le p$, $s<k$ *and* $s(n-k) \;<\; k(n-p)$. *Let $u \in \mathcal{D}^{1,p}(\Omega)$ be a non-negative weak solution of*

the inequality

$$-\ div(|\nabla u|^{p-2}\nabla u)\ \le\ V\frac{|u|^{p-2}}{|x|^s}u \qquad in \quad \Omega, \tag{2.5}$$

i.e.,

$$\int_\Omega |\nabla u|^{\,p-2} < \nabla u, \nabla\phi > \,dz\ \le\ \int_\Omega V\frac{|u|^{p-2}}{|x|^s}u\,\phi\,dz, \tag{2.6}$$

for every $0 \le \phi \in C_o^\infty(\Omega)$.

a) If $V \in L^{r'}(\Omega)$, *then* $u \in L^q\,(\frac{dz}{|x|^t})$ *for any* $0 \le t < \min\{p,s\}$ *and* $q \ge p^*(s)$. *In particular* $u \in L^q(\Omega)$ *for every* $p^* \ \le\ q\ <\ \infty$.

b) If $V \in L^{t_o}(\Omega) \cap L^{r'}(\Omega)$ *for some* $t_o\ >\ r'$, *then* $u \in L^\infty(\Omega)$.

Remark 2.2. *a) As usual, if we have equality instead of inequality in the above Theorem, the conclusion holds for any weak-solution, without a sign condition. The proof below works with very minor changes. This is the reason we use* $|u|$ *rather than* u *when we are dealing with a non-negative function.*

b) The use of the weighted L^q *spaces in part (a) of the Theorem is essential.*

Proof. The assumption $V \in L^{r'}(\Omega)$ together with the Hardy-Sobolev inequality shows that (2.6) holds true for any $\phi \in \mathcal{D}^{\,1,p}(\Omega)$. This can be seen by approximating in the space $\mathcal{D}^{\,1,p}(\Omega)$ by a sequence of test functions $\phi_n \in C_o^\infty(\Omega)$, which will allow to put the limit function in the left-hand side of (2.6) as $|\nabla u|^{p-1} \in L^{p'}$. On the other hand, for $\phi \in C_o^\infty(\Omega)$, using the Hölder and Hardy-Sobolev inequalities we have the estimate

$$\int_\Omega |V|\frac{|u|^{p-1}}{|x|^s}\phi\,dz\ \le\ \Big(\int_\Omega |V|^{r'}\Big)^{1/r'}\Big(\int_\Omega \frac{|u|^{r(p-1)}}{|x|^{rs}}\phi^r\Big)^{1/r} \tag{2.7}$$

$$\le\ \Big(\int_\Omega |V|^{r'}\Big)^{1/r'}\Big[\Big(\int_\Omega \frac{|u|^{rp'(p-1)}}{|x|^{rs}}\Big)^{\frac{1}{rp'}}\Big(\int_\Omega \frac{\phi^{pr}}{|x|^{rs}}\Big)^{1/p}\Big]^{1/r} \tag{2.8}$$

$$\le\ S_p\,\|V\|_{L^{r'}}\Big(\int_\Omega \frac{|u|^{rp'(p-1)}}{|x|^{rs}}\Big)^{\frac{1}{rp'}}\|\nabla\phi\|_{L^p} \quad (\text{ using (2.3) }) \tag{2.9}$$

$$\le\ S_p S_{p,rs}^{p-1}\|V\|_{L^{r'}}\,\|\nabla u\|_{L^p}^{p-1}\,\|\nabla\phi\|_{L^p}, \tag{2.10}$$

which allows to pass to the limit in the right-hand side of (2.6).

We turn to the proofs of a) and b).

a) Let $G(x)$ be a piece-wise smooth, globally Lipschitz function, on the real line, and set

$$F(u)\ =\ \int_0^u |G'(t)|^p\,dt. \tag{2.11}$$

Clearly, F is a non-negative differentiable function with bounded and continuous derivative. From the chain rule, $G(u),\ F(u) \in \mathcal{D}^{\,1,p}(\Omega)$. In particular, $F(u)$ is a legitimate test function in (2.6). We are going to show that if q is a number $q \ge p^*(s)$ and $0 \le t < \min\{p,s\}$, then $u \in L^q\,(\frac{dz}{|x|^s})$ implies $u \in \mathcal{D}^{\,1,p}(\Omega) \cap L^{\kappa_t q}\,(\frac{dz}{|x|^t})$, where $\kappa_t = \frac{p^*(t)}{p}$, and for some positive constant C depending on q we have

$$\|u\|_{L^{\kappa_t q}\,(\frac{dz}{|x|^t})}\ \le\ C\ \|u\|_{L^q\,(\frac{dz}{|x|^s})}. \tag{2.12}$$

Notice that we can apply the Hardy-Sobolev inequality replacing the exponent s with the exponent t. Furthermore, we require $t < p$ as then we have

$$\kappa_t = \frac{p^*(t)}{p} > 1. \tag{2.13}$$

With $\phi = F(u)$, taking into account $F'(u) = |G'(u)|^p$, the left-hand side of (2.5) can be rewritten as

$$\int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla F(u) > dH = \int_\Omega |\nabla G(u)|^p. \tag{2.14}$$

For $q \geq p^*(s)$, hence $q \geq p^*(p) = p$, we define the function $G(t)$ on the real line in the following way,

$$G(t) = \begin{cases} sign\,(t)\,|t|^{\frac{q}{p}} & \text{if } 0 \leq |t| \leq l, \\ l^{\frac{q}{p}-1} t & \text{if } l < |t|. \end{cases} \tag{2.15}$$

From the power growth of G, besides the above properties, this function satisfies also

$$|u|^{p-1}|F(u)| \leq C(q)|G(u)|^p \leq C(q)|u|^q. \tag{2.16}$$

The constant $C(q)$ depends also on p, but this is a fixed quantity for us. At this moment the value of $C(q)$ is not important, but an easy calculation shows that $C(q) \leq C\,q^{p-1}$ with C depending on p. We will use this in part b). For $M > 0$ to be fixed in a moment we estimate the integral in the right-hand side of (2.6) as follows.

$$\begin{aligned} &\int_\Omega |V| \frac{|u|^{p-1}}{|x|^s} F(u)\, dz \\ &= \int_{(|V|\leq M)} |V| \frac{|u|^{p-1}}{|x|^s} F(u)\, dz + \int_{(|V|>M)} |V| \frac{|u|^{p-1}}{|x|^s} F(u)\, dH \\ &\leq C(q) \int_{(|V|\leq M)} |V| \frac{|G(u)|^p}{|x|^s}\, dz + C(q) \Big(\int_{(|V|>M)} |V|^{r'} \Big)^{\frac{1}{r'}} \Big(\int_\Omega \frac{|G(u)|^{pr}}{|x|^{sr}}\, dz \Big)^{\frac{1}{r}} \\ &\leq C(q) \int_{(|V|\leq M)} |V| \frac{|u|^q}{|x|^s}\, dz + C(q) S^p_{p,rs} \Big(\int_{(|V|>M)} |V|^{r'} \Big)^{\frac{1}{r'}} \Big(\int_\Omega |\nabla G(u)|^p\, dz \Big) \\ &\leq C(q) M\, \|u\|^q_{L^q\,(\frac{dz}{|x|^s})} + C(q) S^p_{p,rs} \|V\|_{L^{r'}\,(|V|>M)}\, \|\nabla G(u)\|^p_{L^p}. \end{aligned} \tag{2.17}$$

At this point we fix once and for all the constant M, so that

$$C(q) S^p_{p,rs} \Big(\int_{(|V|>M)} |V|^{r'}\, dH \Big)^{\frac{1}{r'}} \leq \frac{1}{2},$$

which can be done because $V \in L^{r'}$. Putting together (2.14) and (2.17), and using the Hardy-Sobolev inequality, we come to

$$\|G(u)\|^p_{L^{p^*(t)}(\frac{dz}{|x|^t})} \;\le\; C(q)M\;\|u\|^q_{L^q\,(\frac{dz}{|x|^s})} \tag{2.18}$$

By Fatou's theorem we can let l in the definition of G to infinity and obtain

$$\|u\|^q_{L^{\kappa_t q}(\frac{dz}{|x|^t})} \;\le\; C(q)M\;\|u\|^q_{L^q\,(\frac{dz}{|x|^s})}$$

The proof of a) is finished.

b) Let us observe that the assumption $t_o > r'$ implies that $t'_o < r$ and thus $0 < t'_o s < rs$. Therefore, for any $q \ge p^*(s)$ the norm $\|u\|_{L^{qt'_o}(\frac{dz}{|x|^{st'_o}})}$ is finite from part (a). We are going to prove that the $L^q(\frac{dz}{|x|^{t'_o s}})$ norms of u are uniformly bounded by the $L^{q_o}(\frac{dz}{|x|^{t'_o s}})$ norm of u, where $q_o = t'_o p^*(s)$. We shall do this by iteration and find a sequence q_k which approaches infinity as $k \to \infty$.

Let $q \ge p^*(s)$. We use again the function $F(u)$ from part (a) in the weak form (2.6) of our equation. The left-hand side is estimated from below as before, see (2.14). This time, though, we use Hölder's inequality to estimate from above the right-hand side,

$$\begin{aligned}\int_\Omega |V||u|^{p-2}uF(u)\,dH \;&\le\; \|V\|_{L^{t_o}}\,\||u|^{p-1}F(u)\|_{L^{t'_o}} \\ &\le\; \|V\|_{t_o}\,\|C(q)\frac{|G(u)|^p}{|x|^s}\|_{t'_o} \;\le\; C(q)\|V\|_{L^{t_o}}\,\|u\|^q_{L^{qt'_o}(\frac{dz}{|x|^{st'_o}})}.\end{aligned} \tag{2.19}$$

With the estimate from below we come to

$$\|\nabla G(u)\|^p_{L^p} \;\le\; C(q)\,\|V\|_{L^{t_o}}\,\|u\|^q_{L^{qt'_o}(\frac{dz}{|x|^{st'_o}})}.$$

Using the Hardy-Sobolev inequality and then letting $l \to \infty$ we obtain

$$\|u\|^q_{L^{q\frac{p^*(st'_o)}{p}}(\frac{dz}{|x|^{st'_o}})} \;\le\; C\,C(q)\,\|V\|_{L^{t_o}}\,\|u\|^q_{L^{qt'_o}(\frac{dz}{|x|^{st'_o}})}, \tag{2.20}$$

where C is independent of q.

Let $\delta = \dfrac{p^*(st'_o)}{p\,t'_o}$. A small calculation shows that $\delta > 1$ exactly when $t_o > r'$. With this notation we can rewrite (2.20) as

$$\|u\|^q_{L^{\delta qt'_o}(\frac{dz}{|x|^{st'_o}})} \;\le\; \Big[C\,C(q)\Big]^{\frac{1}{q}}\,\|V\|^{\frac{1}{q}}_{L^{t_o}}\,\|u\|^q_{L^{qt'_o}(\frac{dz}{|x|^{st'_o}})}. \tag{2.21}$$

Recall that $C(q) \le Cq^{p-1}$. At this point we define $q_o = p^*(s)t'_o$ and $q_k = \delta^k q_o$, and after a simple induction we obtain

$$\|u\|_{L^{q_k}(\frac{dz}{|x|^{st'_o}})} \;\le\; \Big\{\prod_{j=0}^{k-1}\big[C\,q_j^{p-1}\big]^{\frac{1}{q_j}}\Big\}\|V\|^{\sum_{j=0}^{k-1}\frac{1}{q_j}}_{L^{t_o}}\;\|u\|_{L^{q_o}(\frac{dz}{|x|^{st'_o}})}. \tag{2.22}$$

Let us observe that the right-hand side is finite,

$$\sum_{j=0}^{\infty} \frac{1}{q_j} \ = \ \frac{1}{q_o} \sum_{j=1}^{\infty} \frac{1}{\delta^j} < \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} \frac{\log q_j}{q_j} \ < \ \infty, \tag{2.23}$$

thanks to $\delta \ > \ 1$. Letting $k \to \infty$ we obtain

$$\|u\|_\infty \ \le \ C\,\|u\|_{L^{q_o}(\frac{dz}{|x|^{s t_o'}})}.$$

□

Remark 2.3. *We should keep in mind that the local version of Theorem 2.1 is also valid. In other words, we can replace all the spaces in the statement of the Theorem with their local version. The proof is accomplished in a very similar fashion by introducing a local cut-off function.*

With the above Theorem we turn to the equation, in fact slightly more general equation, satisfied by the extremals of the Hardy-Sobolev inequality.

Theorem 2.4. *Let Ω be an open subset of $\mathbb{R}^n$, which is not necessarily bounded, $1 < p < n$, $0 \le s \le p$, $s < k$ and $s(n-k) \ < \ k(n-p)$. If $R \in L^\infty$ and $u \in \mathcal{D}^{1,p}(\Omega)$ is a weak non-negative solution to equation*

$$-\ div\,(|\nabla u|^{p-2}\nabla u) \ \le \ R(z)\frac{|u|^{p^*(s)-2}u}{|x|^s}u \qquad in \quad \Omega,$$

then $u \in L^\infty(\Omega)$.

Proof. We define $V \ = \ R|u|^{p^*(s)-p}$. From the Hardy-Sobolev inequality we have $u \in L^{p^*(s)}(\Omega)$ and thus $V \in L^{\frac{p^*(s)}{p^*(s)-p}}(\Omega)$. Since $r' \ = \ \dfrac{p^*(s)}{p^*(s)-p}$, part (a) of Theorem 2.1 shows that $u \in L^q\,(\Omega)$ for $p^* \ \le \ q \ < \ \infty$. Therefore $V \ \in \ L^{\frac{q}{p^*-p}}(\Omega)$ for any such q and thus by part (b) of the same Theorem we conclude $u \in L^\infty(\Omega)$. □

In the next Theorem we show that one can lower the exponent p^* in the L^q regularity of u, when $s = 0$. A similar result in the case $p = 2,\ s = 0$ and $R = |u|^{p^*-p}$ was achieved in [LU], see also [BK] and [GL]. We have to overcome some complications due to the more general structure of the equation, the possible singularity, and the lack of monotonicity of the considered operator.

Theorem 2.5. *Let Ω be an open subset of $\mathbb{R}^n$, which is not necessarily bounded, $1 < p < n$, $0 \le s \le p$, $s < k$ and $s(n-k) \ < \ k(n-p)$. Suppose $R \in L^{r'}$ and $V_o \in L^1 \cap L^{r'}$, and in the case $p > 2$ assume R and V_o are non-negative, $R,\ V_o \ge 0$. If u is a non-negative locally bounded weak solution of the equation*

$$-\triangle_p\, u \ \le \ R\frac{|u|^{p-2}}{|x|^s}u \ + \ V_o, \tag{2.24}$$

then $u \in L^q$ for every $\frac{p^*}{p'} \ < q \ \le \ p^*$.

Remark 2.6. *The condition that u is locally bounded holds for example when $V_o \ = \ 0$ and $R \in L^{r'} \cap L^{t_o}$, for some $t_o > r'$ by Theorem 2.1.*

Proof. Let $0 < \theta < 1/p$ be arbitrarily fixed. Our task then is to show that the function $u^{1-\theta} \in L^{p^*}$.

In the first part of the proof we shall exploit the variational structure of the problem in order to construct suitable test functions, which shall be used in the second part of the proof. Suppose $V, g \in L^1 \cap L^{r'}$ are two given functions.

Consider the functional

$$E(v) \;=\; \frac{1}{p}\int_\Omega |\nabla v|^p\,dz \;-\; \frac{1}{p}\int_\Omega V\frac{|v|^p}{|x|^s}\,dz \;-\; \int_\Omega gv\,dz. \tag{2.25}$$

Note that E is coercive when $\|V\|_{L^{r'}(\Omega)}$ is small. Indeed, we have

$$E(v) \;\geq\; \frac{1}{p}\,\|v\|^p_{\mathcal{D}^{1,p}(\Omega)} \;-\; S^{p-1}_{p,rs}\,\frac{1}{p}\,\|V\|_{L^{r'}}\,\|v\|^p_{\mathcal{D}^{1,p}(\Omega)} \;-\; S_{p,0}\,\|g\|_{L^{(p^*)'}}\,\|v\|_{\mathcal{D}^{1,p}(\Omega)},$$

taking into account (2.39) to justify the finiteness of the norm of g. Furthermore, E is weakly lower semi-continuous and $\mathcal{D}^{1,p}(\Omega)$ is a reflexive Banach space. Therefore, provided that V has a small norm, there exists a minimizer, which is a solution of

$$-\triangle_p\, v \;=\; V\frac{|v|^{p-2}}{|x|^s}\,v \;+\; g. \tag{2.26}$$

Moreover, any solution of the above equation satisfies

$$\|v\|_{\mathcal{D}^{1,p}(\Omega)} \;\leq\; S_p^{1/(p-1)}\,(1-S^p_{p,rs}\|V\|_{L^{r'}})^{1/(p-1)}\,\|g\|^{1/(p-1)}_{L^{(p^*)'}}. \tag{2.27}$$

With this in mind, suppose ϵ is a given positive constant. Since $R \in L^{r'}$ we can fix a large $R_o > 0$ and a small $\delta > 0$, such that

$$\int_{\mathbb{R}^n\setminus B_{R_o}} |R|^{r'}\,dz \;\leq\; \frac{1}{2}\,\epsilon \quad \text{and} \quad \int_{\{|x|<2\delta\}} |R|^{r'}\,dz \;\leq\; \frac{1}{2}\,\epsilon. \tag{2.28}$$

Let $\alpha \in C^\infty$ be a function, $0 \leq \alpha \leq 1$, with

$$\operatorname{supp}\,\alpha \subseteq \{|x| < 2\delta\} \cup \{\mathbb{R}^n \setminus B_{R_o}\}, \qquad \alpha \equiv 1 \quad \text{on} \quad \{|x| < \delta\} \cup \{\mathbb{R}^n \setminus B_{2R_o}\}.$$

In particular supp $(1-\alpha) \subset B_{2R_o} \cap \{|x| > \delta\}$, and hence due to the local boundedness of u we have

$$g = V_o \;+\; (1-\alpha)\;R(z)\;\frac{|u|^{p-2}}{|x|^s}\;u \in\; L^1 \cap L^{r'}.$$

For every $k \in \mathbb{N}$ let $\alpha_k \in C^\infty_o(\mathbb{R}^n)$ be a function, $0 \leq \alpha_k \leq 1$, satisfying

$$\operatorname{supp}\,\alpha_k \;\subset B_{2^{k+1}R_o} \setminus \{|x| < \frac{\delta}{2^{k+1}}\} \quad \text{with} \quad \alpha_k = 1 \quad \text{on} \quad B_{2^kR_o} \setminus \{|x| < \frac{\delta}{2^k}\}.$$

Notice that $\alpha_k \nearrow 1$ a.e. as $k \to \infty$. Define $V = \alpha\,R(z)$ and $V_k = \alpha\,\alpha_k\,R(z)$. Using the properties of the cut-offs we see that these functions enjoy the following properties

$$V \in L^{r'}, \qquad V_k \;\in\; L^1 \cap L^{r'} \quad \text{as} \quad \operatorname{supp}\,V_k \subset\subset \mathbb{R}^n,$$

and

$$V_k \nearrow V \text{ as } k \to \infty.$$

In addition, since

$$\int_\Omega |V|^{r'}\,dz \;\leq\; \int_\Omega \alpha\,|R|^{r'}\,dz \;\leq\; \epsilon,$$

the functions V and hence V_k have small $L^{r'}$ norms, which can be made less than ϵ by taking R_o sufficiently large and δ sufficiently small in (2.28). From now on we assume that R_o and δ have been fixed in the above described manner so that E is coercive and further so that $\|V\|_{L^{r'}} \;\leq\; \epsilon$ with ϵ to be fixed later independently of k, and in fact depending only on $p,\ u,\ \|R\|_{L^\infty}$ and the Sobolev constant.

Rewriting the equation given in the Theorem, we have that u is a given non-negative solution of

$$-\text{div}\,(|\nabla u|^{p-2}\nabla u) \;\leq\; V\frac{|u|^{p-2}}{|x|^s}\,u \;+\; g. \tag{2.29}$$

For every $k \in \mathbb{N}$ let u_k be a solution of

$$-\text{div}\,(|\nabla u_k|^{p-2}\nabla u_k) \;=\; V_k\frac{|u_k|^{p-2}}{|x|^s}\,u_k \;+\; g. \tag{2.30}$$

Moreover, when $p > 2$ we assume that $R,\ V_o \geq 0$ and thus if we take u_k to be a minimizer of

$$E_k(v) \;=\; \frac{1}{p}\int_\Omega |\nabla v|^p\,dz \;-\; \frac{1}{p}\int_\Omega V_k\frac{|v|^p}{|x|^s}\,dz \;-\; \int_\Omega gv\,dz. \tag{2.31}$$

then u_k is non-negative, i.e., when $p > 2$ we have $u_k \geq 0$. This shall be used at the very end of the proof.

Next we define the necessary cut-off, which shall be used in the final step. Let $\eta_m(t)$ be the following function

$$\eta_m(t) \;=\; \begin{cases} t, & t > 1/m \\ \\ m^{\frac{\theta p}{1-p\,\theta}}\;t^{\frac{1}{1-p\,\theta}}, & 0 \leq\; t \leq \frac{1}{m}., \end{cases} \tag{2.32}$$

Observe that η_m is a continuous function and

$$0 \leq \eta_m' \leq \max\{1, \frac{1}{1-p\,\theta}\} \;=\; \frac{1}{1-p\,\theta},$$

which implies the useful fact

$$\eta_m'(t) \;\leq\; \frac{1}{1-p\,\theta}\;t.$$

We define also

$$\phi_m(t) \;=\; \eta_m^{1-p\theta} \;\;\text{and}\;\; f_m(t) \;=\; \eta_m^{1-\theta}. \tag{2.33}$$

A short calculation gives

$$\phi_m(t) \;=\; \begin{cases} t^{1-p\theta}, & t > 1/m \\ \\ m^{p\theta}t, & t \leq \frac{1}{m} \end{cases} \qquad\qquad \phi_m'(t) \;\leq\; \begin{cases} (1-p\,\theta)\;m^{p\theta}, & t > 1/m \\ \\ m^{p\theta}, & t \leq \frac{1}{m} \end{cases} \tag{2.34}$$

In particular, for every fixed m, we have that ϕ_m' is a bounded function and thus if $0 \leq\; v \in \mathcal{D}^{1,p}(\Omega)$ then $0 \leq \phi_m(v) \in \mathcal{D}^{1,p}(\Omega)$. Since $1-\theta \;>\; 1-p\theta$ the derivative $f_m'(t)$ is also bounded and thus $f_m(v) \,\in \mathcal{D}^{1,p}(\Omega)$. From now on, for simplicity, given a function $0 \leq\, v \in \mathcal{D}^{1,p}(\Omega)$ we let

$$\eta = \eta_m(v), \qquad\qquad \phi = \phi_m(v) \qquad \text{and} \qquad f = f_m(v),$$

all of which, due to the chain rule, are functions from $\mathcal{D}^{1,p}(\Omega)$.

A small but very important calculation shows that we have

$$|\nabla f|^p \;=\; (1-p\,\theta)^{p-1}\Big(\frac{1-\theta}{1-p\theta}\Big)^p\,|\nabla\eta|^{p-2}\,\nabla\eta\cdot\nabla\phi.$$

With $c_\theta \;=\; \left(\frac{1-\theta}{1-p\,\theta}\right)^p$, using $\nabla v\cdot\nabla\phi \geq 0$ and the above identity, we compute

$$\int_\Omega |\nabla\eta|^{p-2}\,\nabla\eta\cdot\nabla\phi\,dz \;=\; \int_{\{v<\frac{1}{m}\}}|\nabla\eta|^{p-2}\,\nabla\eta\cdot\nabla\phi\,dz \;+\; \int_{\{\frac{1}{m}<v\}}|\nabla\eta|^{p-2}\,\nabla\eta\cdot\nabla\phi\,dz$$

$$= \frac{m^{(p-1)p\,\theta/(1-p\,\theta)}}{(1-p\,\theta)^{p-1}}\int_{\{v<\frac{1}{m}\}} v^{(p-1)p\,\theta/(1-p\,\theta)}\;|\nabla u|^{p-2}\nabla v\cdot\nabla\phi\,dz$$

$$+\int_{\{\frac{1}{m}<v\}}|\nabla v|^{p-2}\,\nabla v\cdot\nabla\phi\,dz$$

$$\le \frac{1}{(1-p\,\theta)^{p-1}}\int_{\{v<\frac{1}{m}\}}|\nabla u|^{p-2}\,\nabla v\cdot\nabla\phi\,dz \;+\; \int_{\{\frac{1}{m}<v\}}|\nabla v|^{p-2}\,\nabla v\cdot\nabla\phi\,dz$$

$$\le \int_\Omega|\nabla v|^{p-2}\,\nabla v\cdot\nabla\phi\,dz \quad \text{as} \quad 1\;\le\;1/(1-p\,\theta)$$

Therefore the following bound holds

$$\int_\Omega|\nabla f|^p\,dz \;\le\; c_\theta\int_\Omega|\nabla v|^{p-2}\,\nabla v\cdot\nabla\phi\,dz. \tag{2.35}$$

Let us set

$$v=u_k^+$$

and use ϕ as a test function in (2.30). The inequality above gives an estimate of the left-hand side. For the first term on the other side we have

$$\int_\Omega V_k\frac{|v|^{p-1}}{|x|^s}\,\phi\,dz \;=\; \int_{\{v>1/m\}}V_k\frac{|v|^{p-1}}{|x|^s}\,\phi\,dz \;+\; \int_{\{v<\frac{1}{m}\}}V_k\frac{|v|^{p-1}}{|x|^s}\,\phi\,dz$$

$$\le\; S_p^p\int_\Omega V_k\frac{|v|^{p-1+1-\theta p}}{|x|^s}\,dz \;+\; m^{\theta p}\int_{\{v<\frac{1}{m}\}}V_k\frac{|v|^p}{|x|^s}\,dz$$

$$\le\; S_p^p\|V_k\|_{L^{r'}}\int_\Omega|\nabla f|^p\,dz \;+\; m^{\theta p-p}\,\frac{2^{s(k+1)}}{\delta^s}\,\|V_k\|_{L^1}$$

$$\le\; S_p^p\|V\|_{L^{r'}}\int_\Omega|\nabla f|^p\,dz \;+\; m^{\theta p-p}\,\frac{2^{s(k+1)}}{\delta^s}\,\|V_k\|_{L^1}$$

using $pr=p^*(rs)$ and the fact that $f\in\mathcal{D}^{1,p}(\Omega)$. The other term on the right-hand side can be estimated in the following way

$$\int_\Omega g\phi\, dz \;\le\; \int_{\{v<\frac{1}{m}\}} |g|\;\phi\, dz \;+\; \int_{\{\frac{1}{m}<v\}} |g|\;\phi\, dz$$
$$\le\; m^{p\,\theta} \int_{\{v<\frac{1}{m}\}} |g|\; v\, dz \;+\; \int_{\{\frac{1}{m}<v\}} |g|\; v^{1-\theta p}\, dz$$
$$\le\; m^{\theta p-1} \|g\|_{L^1} \;+\; \|g\|_{L^q}\; \|v\|^{p^*/q}_{L^{p^*}}, \tag{2.36}$$

where

$$q \;=\; \Big(\frac{p^*}{1-p\theta}\Big)' \;=\; \frac{p^*}{p^*-(1-p\theta)} \;>\; 1, \tag{2.37}$$

as $1-p\,\theta>0$. Recall we are assuming only $g\in L^1\cap L^{r'}$. The definition (2.37) of q shows that $q<(p^*)'$, while

$$p^*(p)=p \;\le\; p^*(s) \;\le\; p^*(0)=p^* \tag{2.38}$$

and thus

$$r' = \frac{p^*}{p^*(s)-p} \;\ge\; \frac{p^*}{p^*-p} \;=\; \big(p^*/p\big)' \;>\; \frac{p^*}{p^*-1} \;=\; (p^*)'. \tag{2.39}$$

Therefore we have

$$q \;<\; r'$$

and $g\in L^q_{\rm loc}$ for every $1\le q<r'$.

Putting the above three estimates together we have shown that the gradient of f satisfies the inequality (with $S_p \;=\; S_{p,o}$)

$$\frac{1}{c_\theta}\int_\Omega |\nabla f|^p\, dz \;\le\; S_p^p \|V\|_{L^{r'}} \int_\Omega |\nabla f|^p\, dz \;+\; m^{\theta p-p}\; \frac{2^{s(k+1)}}{\delta^s}\; \|V_k\|_{L^1}$$
$$+\; m^{\theta p-1}\|g\|_{L^1} +\; \|g\|_{L^q(v>1/m)}\; \|v\|^{p^*/q}_{L^{p^*}(v>1/m)}. \tag{2.40}$$

With the above estimates at hand, we can conclude by moving the first term from the right side to the left, and then letting $m\to\infty$, provided we have

$$1-c_\theta S_p^p \|V\|_{L^{r'}} > 0, \text{ i.e.,} \quad \|V\|_{L^{r'}} \ll \tag{2.41}$$

Taking the initial R_o sufficiently large at the start of the proof, we can let $m\to\infty$, recall $f_m\to(v^+)^{1-\theta}$, which gives the inequality

$$\int_\Omega |\nabla (v^+)^{1-\theta}|^p\, dz \;\le\; C_{p,\theta} \|g\|_{L^q(\Omega)}\; \|v\|^{p^*/q}_{L^{p^*}(\Omega)},$$

Working similarly with v^- we can prove eventually, recall $v=u_k$, that the $\mathcal{D}^{1,p}(\mathbb{R}^n)$ norms of $u_k^{1-\theta}$ satisfy

$$\|u_k^{1-\theta}\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n)} \;=\; C_{p,\theta}\, \|g\|_{L^q(\Omega)}\; \|u_k\|^{p^*/q}_{L^{p^*}(\Omega)} \tag{2.42}$$

and hence they are uniformly bounded in view of (2.27).

The proof is finished by letting $k \to \infty$, but the argument in the cases $p=2$ and $p>2$ are different. When $p>2$ we take into account that by construction $u_k \geq 0$, cf. the line after (2.30), and if $u_k \to \bar{u}$ weakly in $\mathcal{D}^{1,p}(\Omega)$, where

$$-\triangle_p\, \bar{u} \;=\; V\frac{\bar{u}^{p-1}}{|x|^s} \;+\; g,$$

from Lemma 2.7 we conclude $u^{1-\theta} \in L^{p^*}$. If $p=2$ the situation is simpler since we can use the monotonicity. In fact, since $\|V\|_{L^{r'}}$ is small and $u,\ \bar{u} \in \mathcal{D}^{1,p}(\Omega)$ Hölder's inequality and the strong monotonicity of the laplacian give $(u-\bar{u})^+ = 0$. Indeed, using $w = (u-\bar{u})^+$ as a test function in the inequality

$$-\triangle u \;+\; \triangle \bar{u} \;\leq\; V\frac{u-\bar{u}}{|x|^s}$$

we see that, cf. (2.3),

$$\|w\|^2_{\mathcal{D}^{1,2}(\Omega)} \leq C\ \|V\|_{L^{r'}}\ \Big(\int \frac{w^{2r}}{|x|^{rs}}\,dz\Big)^{1/r} \;\leq\; C\ \|V\|_{L^{r'}}\ \|w\|^2_{\mathcal{D}^{1,2}(\Omega)} \;\leq\; \frac{1}{2}\ \|w\|^2_{\mathcal{D}^{1,2}(\Omega)},$$

hence $\|w\|_{\mathcal{D}^{1,2}(\Omega)} = 0$ and thus $w=0$, i.e., $u \leq \bar{u}$. The proof is complete. □

In the proof of the previous Theorem we used the following comparison/uniqueness principle for the p-laplacian on an unbounded domain.

Lemma 2.7. *Let $V \in L^{r'}$ and $u,\ \bar{u} \in \mathcal{D}^{1,p}(\Omega)$, $u \geq 0$, be weak solutions, respectively, of*

$$-\triangle_p\, u \;\leq\; V\frac{u^{p-1}}{|x|^s} \;+\; g \tag{2.43}$$

$$-\triangle_p\, \bar{u} \;=\; V\frac{|\bar{u}|^{p-2}\bar{u}}{|x|^s} \;+\; g. \tag{2.44}$$

a) Suppose $\bar{u} \geq 0$ and $g \geq 0$. If $g \neq 0$ then $u \;\leq\; \bar{u}$. Otherwise, $u \;=\; c\,\bar{u}$ on the set $\{u \geq \bar{u}\}$.
b) More generally, suppose $g\,\bar{u} \geq 0$ and $\bar{u}$ is a super solution of (2.44)*, i.e.,*

$$-\triangle_p\, \bar{u} \;\geq\; V\frac{|\bar{u}|^{p-2}\bar{u}}{|x|^s} \;+\; g.$$

If $g\,\bar{u} \not\equiv 0$ then $u \;\leq\; |\bar{u}|$. Otherwise, $u \;=\; c\,|\bar{u}|$ on the set $\{u \geq |\bar{u}|\}$.

Proof. a) For ease of reading let us consider first the case when equality holds in the inequality satisfied for u, i.e., suppose $u,\ \bar{u} \in \mathcal{D}^{1,p}(\Omega)$ are two non-negative weak solutions. We are going to show that if $g \neq 0$ then $u \;=\; \bar{u}$. Otherwise, one of the solutions is a constant multiple of the other. Working as in Lemma 3.1 in [Lin], see also [DS] and [A], we define

$$u_\epsilon \;=\; u \;+\; \epsilon \quad \text{ and } \quad \bar{u}_\epsilon \;=\; \bar{u} \;+\; \epsilon$$

and the two test function

$$\eta \;=\; \frac{{u_\epsilon}^p - \bar{u}_\epsilon^p}{{u_\epsilon}^p} \;=\; u_\epsilon \;-\; \big(\frac{\bar{u}_\epsilon}{u_\epsilon}\big)^p\, u_\epsilon \quad \text{ and } \quad \bar{\eta} \;=\; \frac{\bar{u}_\epsilon^p - {u_\epsilon}^p}{\bar{u}_\epsilon^p} \;==\; \bar{u}_\epsilon \;-\; \big(\frac{u_\epsilon}{\bar{u}_\epsilon}\big)^{p-1}\, \bar{u}_\epsilon.$$

Multiplying the equation for u by η and the equation for $\bar{u}$ by $\bar{\eta}$, and then adding the two equations we have

$$\begin{aligned}(2.45)\quad &\int_\Omega |\nabla u|^{p-2}\ \nabla u\cdot\nabla\eta\ +\ |\nabla\bar u|^{p-2}\ \nabla\bar u\cdot\nabla\bar\eta\, dz\\ &\qquad = \int_\Omega \frac{V}{|x|^s}\ \left(u^{p-1}\eta\ +\ \bar u^{p-1}\bar\eta\right) dz\ +\ \int_\Omega g\ (\eta\ +\ \bar\eta)\, dz.\end{aligned}$$

Using

$$\nabla\eta = \left[1\ +\ (p-1)\Big(\frac{\bar u_\epsilon}{u_\epsilon}\Big)^p\right]\ \nabla u_\epsilon\ -\ p\ \Big(\frac{\bar u_\epsilon}{u_\epsilon}\Big)^{p-1}\ \nabla\bar u_\epsilon$$

we find

$$\begin{aligned}|\nabla u|^{p-2}\ \nabla u\cdot\nabla\eta\ &=\ \left[1\ +\ (p-1)\Big(\frac{\bar u_\epsilon}{u_\epsilon}\Big)^p\right]\ |\nabla u_\epsilon|^p\ -\ p\ \Big(\frac{\bar u_\epsilon}{u_\epsilon}\Big)^{p-1}\ |\nabla u_\epsilon|^{p-2}\ \nabla u_\epsilon\cdot\nabla\bar u_\epsilon\\
&= \Big({u_\epsilon}^p\ +\ (p-1)\ \bar u_\epsilon^p\Big)\ |\nabla\ln u_\epsilon|^p\ -\ p\ \bar u_\epsilon^p\ |\nabla\ln u_\epsilon|^{p-2}\ \langle\nabla\ln u_\epsilon,\ \nabla\ln\bar u_\epsilon\rangle\\
&=\ {u_\epsilon}^p\ |\nabla\ln u_\epsilon|^p\ +\ \bar u_\epsilon^p\ \Big[-|\nabla\ln u_\epsilon|^p\ -\ p\ |\nabla\ln u_\epsilon|^{p-2}\ \langle\nabla\ln u_\epsilon,\ \nabla\ln\bar u_\epsilon-\nabla\ln u_\epsilon\rangle\Big]\\
&\geq\ {u_\epsilon}^p\ |\nabla\ln u_\epsilon|^p\ +\ \bar u_\epsilon^p\ \Big[\ C_p\ |\nabla\ln u_\epsilon\ -\ \nabla\ln\bar u_\epsilon|^p - |\nabla\ln\bar u_\epsilon|^p\ \Big]\\
&=\ {u_\epsilon}^p\ |\nabla\ln u_\epsilon|^p\ -\ \bar u_\epsilon^p\ |\nabla\ln\bar u_\epsilon|^p\ +\ C_p\ \bar u_\epsilon^p\ |\nabla\ln u_\epsilon\ -\ \nabla\ln\bar u_\epsilon|^p,\end{aligned}$$

using the inequality, cf. Lemma 4.2 of [Lin],

$$|a|^p\ >\ |b|^p\ +\ p\ |b|^{p-2}\ \langle b,\ a\ -\ b\rangle + C_p\ |a\ -\ b|^p\ ,\quad a,\ b\in\mathbb{R}^n,\quad a\neq b.$$

The inequality we just proved shows the left-hand side of (2.45) can be estimated as follows

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\ \nabla u\cdot\nabla\eta\ +\ |\nabla\bar u|^{p-2}\ \nabla\bar u\cdot\nabla\bar\eta\, dz\\ &\qquad\geq\ C_p\ \int_\Omega ({u_\epsilon}^p\ +\ \bar u_\epsilon^p)\ |\nabla\ln u_\epsilon\ -\ \nabla\ln\bar u_\epsilon|^p\, dz.\end{aligned}$$

Therefore we have

$$\begin{aligned}C_p\int({u_\epsilon}^p+\bar u_\epsilon^p)\ |\nabla\ln u_\epsilon-\nabla\ln\bar u_\epsilon|^p\, dz\ &\leq\ \int\frac{V}{|x|^s}\Big[\Big(\frac{u}{u_\epsilon}\Big)^{p-1}\ -\ \Big(\frac{\bar u}{\bar u_\epsilon}\Big)^{p-1}\Big]\ ({u_\epsilon}^p-\bar u_\epsilon^p)\, dz\\ &\quad +\ \int g\ \bar u_\epsilon\ \Big[\frac{u_\epsilon}{\bar u_\epsilon}+1-\Big(\frac{u_\epsilon}{\bar u_\epsilon}\Big)^p\ -\Big(\frac{u_\epsilon}{\bar u_\epsilon}\Big)^{1-p}\ \Big]\, dz.\end{aligned}$$

The first integral on the right goes to 0 when $\epsilon\to 0$ by the Lebesgue convergence theorem since $V\in L^{r'}$. A small calculation shows that the function

$$f(t)\ =\ 1\ +\ t\ -\ t^p\ -\ t^{1-p}$$

is negative for $t>0$ with $f=0$ iff $t=1$. In fact $f(1)=0$, $f''(t)<0$ and $f'(1)=0$. The proof of the case when we have equality in both places is complete.

Now, consider the more general case. Notice that $\eta\geq 0$ iff $\bar\eta\leq 0$. Therefore we can multiply the inequality satisfied by u with the test function η^+ and then add the equation satisfied by $\bar u$

after we multiply it with $\bar\eta^- = \min\{\bar\eta, 0\}$. Then we work as before, but this time the conclusions will hold only on the set $\{u \geq \bar u\}$. The proof of part (a) is complete.

b) Define $v = \bar u^+ = \max\{\bar u, 0\}$, hence $0 \leq v \in \mathcal{D}^{1,p}(\Omega)$, $v_\epsilon = v + \epsilon$ and consider

$$\bar\eta \; = \; \frac{{v_\epsilon}^p - {u_\epsilon}^p}{{v_\epsilon}^p} \; = = \; v_\epsilon \; - \; (\frac{u_\epsilon}{v_\epsilon})^p \, v_\epsilon. \tag{2.46}$$

Therefore we can multiply the inequality satisfied by u with the test function η^+ and then add the inequality satisfied by $\bar u$ after we multiply it with $\bar\eta^- \leq 0$, which will bring us to

$$C_p \int ({u_\epsilon}^p + {v_\epsilon}^p) \; |\nabla \ln u_\epsilon - \nabla \ln v_\epsilon|^p \, dz \; \leq \; \int \frac{V}{|x|^s} \Big[\Big(\frac{u}{u_\epsilon}\Big)^{p-1} \; - \; \Big(\frac{v}{v_\epsilon}\Big)^{p-1} \Big] \, ({u_\epsilon}^p - {v_\epsilon}^p) \, dz$$

$$+ \; \int g \; v_\epsilon \;\; f(u_\epsilon/v_\epsilon) dz.$$

Since $g\,\bar u \geq 0$, letting $\epsilon \to 0$ we see that with $v = \bar u^+$ we have $u \leq v$ if $g\,\bar u > 0$ somewhere, and u and v are proportional on the set $\{u \geq v\}$ otherwise, i.e., in the case $g\,\bar u \equiv 0$. Finally, let us observe that $-\bar u$ is a solution of

$$-\triangle_p \, (-\bar u) \; \geq \; V \frac{|\bar u|^{p-2}(-\bar u)}{|x|^s} \; - \; g.$$

Taking $v = \max\{-\bar u, 0\} = \; -\bar u^-$ and observing that $-g v_\epsilon \; = \; g\bar u^- \; - \; \epsilon\, g$ the argument above shows that $u \leq v$ if $g\,\bar u > 0$ somewhere, and u and v are proportional on the set $\{u \geq v\}$ otherwise, i.e., in the case $g\,\bar u \equiv 0$. The conclusion is that if $g\,\bar u \not\equiv 0$ then

$$u \; \leq \; |\bar u|,$$

while if $g\,\bar u \equiv 0$ then $u = c\bar u$ on the set $u \geq |\bar u|$. □

So far we have concerned ourselves with the global properties of solutions. Having done this we can obtain the asymptotic behavior of solutions at infinity. The first result concern the local behavior on a ball away from some finite point, see [E] and [Z] for related results.

Theorem 2.8. *Suppose s, p, k, and n satisfy the conditions of Theorem 2.1. Let u be a nonnegative solution to the inequality* (2.5)*, with $V \in L^{r'}$. We assume that u has been extended with zero outside Ω. Suppose that $q_o \geq p$ is an exponent such that $u \in L^{q_o}(\Omega)$. There exist constants $C = C(\mathbb{R}^n, p, ||u||_{\mathcal{D}^{1,p}(\Omega)}, \|u\|_{q_o}) > 0$ and $0 < R_o = R_o(\|V\|_{L^{r'}})$, such that, for every $z \in \mathbb{R}^n$ and $R = |z|/2 \geq R_o$ we have*

$$\max_{B(z,R/2)} u \leq C \left(\frac{1}{B(z,R)} \int_{B(z,R)} u^{q_o} dx \right)^{\frac{1}{q_o}}. \tag{2.47}$$

Furthermore, u has the following decay at infinity

$$u(z) \; \leq \; \frac{C}{|z|^{n/q_o}}. \tag{2.48}$$

Proof. Given a function $\alpha \in C_o^\infty(\mathbb{R}^n)$, $\alpha \geq 0$ we consider the function $\alpha^p F(u) \in \mathcal{D}^{1,p}(\Omega)$, see (2.11) and (2.15). We recall (2.16), and observe in addition

$$F(u) \; \leq \; |G'(u)|^p. \tag{2.49}$$

We use the fact $q/p > 1$ in order to see that G is a piece-wise smooth and globally Lipschitz function. Using $\alpha^p F(u)$ as a test function in the weak formulation (2.6) we have

$$\int_\Omega |\nabla u|^{p-2} < \nabla u, \nabla(\alpha^p F(u)) > dz \;\le\; \int_\Omega V \frac{|u|^{p-2}}{|x|^s} u\, \alpha^p F(u)\, dz$$

Let us consider the left-hand side (LHS) of the above inequality, which is easily seen to equal

$$\text{LHS} \;=\; \int \alpha^p |\nabla G|^p\, dz \;+\; p\int |\nabla u|^{p-2}\alpha^{p-1} F(u)\nabla u\cdot\nabla\alpha\, dz. \tag{2.50}$$

For any $\epsilon > 0$ we have $ab \le \epsilon\frac{a^p}{p} + \epsilon^{-p'/p}\frac{b^{p'}}{p'}$, and hence

$$\begin{aligned} |\nabla u|^{p-2}\alpha^{p-1}F(u)\nabla u\cdot\nabla\alpha \;&\le\; \frac{\epsilon}{p}\alpha^p|\nabla u|^p u^{-1}F(u) \;+\; \frac{\epsilon^{-p'/p}}{p'}|\nabla\alpha|^p F(u) u^{p/p'} \\ &\le\; \frac{\epsilon}{p}\alpha^p|\nabla G|^p \;+\; C\epsilon^{-1}q^{p-1}|\nabla\alpha|^p G^p, \end{aligned} \tag{2.51}$$

after using (2.49) and (2.16) in the last inequality. Inserting (2.51) in (2.50) we find

$$\text{LHS} \;\ge\; (1-\epsilon)\int \alpha^p|\nabla G|^p\, dz \;-\; C\epsilon^{-p'/p}q^{p-1}\int |\nabla\alpha|^p G^p\, dz.$$

We shall use the above inequality with a fixed sufficiently small ϵ so that $1-\epsilon > 0$. With the help of $(a+b)^p \le 2^p(a^p+b^p)$, and using once more (2.16) and the paragraph after it, we come to

$$\int |\nabla(\alpha G)|^p\, dz \;\le\; C\int_\Omega V\frac{\alpha^p G^p}{|x|^s}\, dz \;+\; Cq^{p-1}\int |\nabla\alpha|^p G^p\, dz, \tag{2.52}$$

where C is a constant independent of q. Therefore, using the Hölder and the Hardy-Sobolev inequalities ($pr = p^*(rs)!$), we have

$$\|\nabla(\alpha G)\|^p_{L^p} \;\le\; C\|V\|_{L^{r'}(\operatorname{supp}\alpha)}\|\nabla(\alpha G)\|^p_{L^p} \;+\; Cq^{p-1}\int |\nabla\alpha|^p G^p\, dz \tag{2.53}$$

Since $V \in L^{r'}$ it follows that if $0 \ne z \in \mathbb{R}^n$ and $R = |z|/2 \ge R_o$ then

$$\int_{B_R(z)} V^{r'}\, dz \;\to 0 \qquad \text{as} \qquad R_o \to \infty. \tag{2.54}$$

Therefore, $C\|V\|_{L^{r'}(\operatorname{supp}\alpha)} \le 1/2$ for all α with

$$\alpha \in C^\infty_o\big(B_R(z)\big) \qquad \text{with} \qquad R = |z|/2, \qquad |z| \ge R_o, \tag{2.55}$$

where R_o depends on V and C, and shall be fixed for the rest of the proof. Using the Sobolev inequality, we have shown that for any such α we have

$$\|\alpha G\|_{L^{p^*}} \;\le\; \|\nabla(\alpha G)\|_{L^p} \;\le\; Cq^{(p-1)/p}\|(\nabla\alpha)G\|_{L^p} \tag{2.56}$$

Therefore, if $u \in L^q$ we can apply Fatou's theorem when $l \to \infty$ to get

$$\Big(\int \alpha^{p^*}u^{\delta q}\Big)^{1/p^*} \;\le\; C\; q^{(p-1)/p}\Big(\int |\nabla\alpha|^p u^q\Big)^{1/p}, \tag{2.57}$$

where $\delta = p^*/p > 1$. In particular, for any $0 < \rho < r < R$ and $\alpha \in C^\infty_o\big(B(z,r)\big)$ with $\alpha \equiv 1$ on $B(z,\rho)$ and $|\nabla\alpha| \le \frac{2}{r-\rho}$, we have

$$\Big(\frac{1}{|B(z,\rho)|}\int_{B(z,\rho)} u^{\delta q}\, dz\Big)^{\frac{1}{\delta q}} \;\le\; \frac{C^{p/q}q^{(p-1)/q}}{(r-\rho)^{p/q}}\Big(\frac{1}{|B(z,r)|}\int_{B(z,r)} u^q\, dz\Big)^{\frac{1}{q}}. \tag{2.58}$$

We can define the sequences $q_j = q_o\delta^j$ and $r_j = \frac{R}{2}\big(1+\frac{1}{2^j}\big)$ for $j = 0, 1, \ldots$ with which Moser's iteration procedure gives inequality (2.47). Let us observe that $\sum_{j=0}^{\infty}\frac{1}{q_j} < \infty$ and $\sum_{j=0}^{\infty}\frac{\ln q_j}{q_j} < \infty$ thanks to $\delta > 1$, cf. (2.23). The decay property follows immediately taking into account that the volume of $B_R(z)$ is proportional to R^n, i.e., $|z|^n$. □

Combining Theorems 2.1, 2.5 and (2.48), we can assert the following decay of solutions to (2.5).

Theorem 2.9. *Let Ω be an open subset of $\mathbb{R}^n$, which is not necessarily bounded, $1 < p < n$, $0 \le s \le p$, $s < k$ and $s(n-k) \ < \ k(n-p)$. Suppose $R \in L^{r'} \cap L^{t_o}$, for some $t_o > r'$, and in the case $p > 2$ assume that R and V_o are non-negative, $R \ge 0$, $V_o \ge 0$.*
If u is a non-negative solution of (2.24) then there exists a $C = C(\mathbb{R}^n, p, ||u||_{\mathcal{D}^{1,p}(\Omega)}) > 0$, such that, u has the following decay at infinity

$$u(z) \ \le \ \frac{C}{|z|^q}\,\|u\|_{D^{1,p}(\Omega)}, \tag{2.59}$$

for any $q < \frac{n-p}{p-1}$.

Remark 2.10. *Let us observe that the fundamental solution of the p-laplacian on $\mathbb{R}^n$ equals $C|z|^{-\frac{n-p}{p-1}}$, where C is a constant.*

3. Asymptotics for the scalar curvature equation

In this section we restrict our considerations to the case $p = 2$ and furthermore we require $V = R(z)u^{2^*-2}$ with $R \in L^\infty$, i.e., we consider a non-negative weak solution u of

$$-\triangle\, u \ \le \ \frac{R(z)u^{2^*(s)-1}}{|x|^s} \qquad \text{in} \quad \Omega. \tag{3.1}$$

From Theorem 2.9 we know that for any $0 < \theta < 1$ there exists a constant $C_\theta > 0$, such that,

$$u(z) \ \le \ \frac{C_\theta}{1+|z|^{\theta(n-2)}}\,\|u\|_{D^{1,p}(\Omega)}. \tag{3.2}$$

The next result shows that the decay is at least as that of the fundamental solution. Furthermore, if u is a non-negative solution, rather than a subsolution then u has the same decay as the fundamental solution.

Theorem 3.1. *Let Ω be an open subset (not necessarily bounded) of $\mathbb{R}^n$, $n > 2$, $0 \le s \le 2$, $s < k$ and $R \in L^\infty$.*
a) If u is a non-negative solution of (3.1) then there exists a constant $C > 0$, such that,

$$0 \ \le \ u(z) \ \le \ \frac{C}{1+|z|^{n-2}}, \quad z \in \Omega. \tag{3.3}$$

b) If u is a non-negative non-trivial solution of

$$-\triangle\, u \ = \ \frac{R(z)u^{2^*(s)-1}}{|x|^s} \qquad \text{in} \quad \Omega, \tag{3.4}$$

with $R \ge 0$ then

$$\frac{C^{-1}}{1+|z|^{n-2}} \ \le \ u(z) \ \le \ \frac{C}{1+|z|^{n-2}}, \quad z \in \Omega. \tag{3.5}$$

Proof. We shall prove first the estimate for u from above. Let us extend u as a function on $\mathbb{R}^n$ be setting it equal to zero outside of Ω. Let us define $f = \frac{|R(z)|u^{2^*-1}}{|x|^s}$. Thanks to Theorem 2.9 we have for any $0<\theta<1$

$$f(z) \ \leq \ C_\theta \frac{|R(z)|}{1+|z|^{\theta(n+2-s)}}\frac{1}{|x|^s}. \tag{3.6}$$

Consider the function $v = \Gamma * f$, where Γ is the positive fundamental solution of the laplacian $\Gamma(z) = \frac{1}{n(n-2)\omega_n}|z|^{2-n}$ (recall $n>2$), where ω_n is the volume of the unit ball in $\mathbb{R}^n$, so that $-\triangle\Gamma = \delta$. For points z, ζ in $\mathbb{R}^n$ we shall write $z=(x,y)\in R^k\times\mathbb{R}^{n-k}$ and $\zeta = (\xi,\eta)\in R^k\times\mathbb{R}^{n-k}$. With this notation and using (3.6) we have ($C_n = \frac{1}{n(n-2)\omega_n}$)

$$\begin{aligned}
v(z) \ &= \ C_n \int_{\mathbb{R}^n} \frac{f(\zeta)}{|z-\zeta|^{n-2}}\, d\zeta \\
&= \ C_n \int_{|z-\zeta|\leq\frac{|z|}{2}} \frac{f(\zeta)}{|z-\zeta|^{n-2}}\, d\zeta \\
&\qquad + \ C_n \int_{|z-\zeta|>\frac{|z|}{2}} \frac{f(\zeta)}{|z-\zeta|^{n-2}}\, d\zeta \\
&\qquad \leq \ \frac{C_\theta\|R\|_\infty}{1+|z|^{\theta(n+2-s)}} \int_{|z-\zeta|\leq\frac{|z|}{2}} \frac{1}{|z-\zeta|^{n-2}}\frac{1}{|\xi|^s}\, d\zeta \\
&\qquad + \ \frac{C_\theta\|R\|_\infty}{1+|z|^{n-2}} \int_{\mathbb{R}^k}\int_{\mathbb{R}^{n-k}} \frac{1}{(1+|\zeta|^{\theta(n+2-s)})}\frac{1}{|\xi|^s}\, d\eta\, d\xi \\
&\overset{def}{\equiv} \ \frac{C_\theta\|R\|_\infty}{1+|z|^{\theta(n+2-s)}}\, I_1 \ + \ \frac{C_\theta\|R\|_\infty}{1+|z|^{n-2}}\, I_2 \ \leq \ \frac{C_\theta\|R\|_\infty}{1+|z|^{n-2}},
\end{aligned}$$

from Lemmas 3.5 and 3.3 (see also the Remark following Lemma 3.5 for the case $k=n$).

Going back to the bound from above for u we see that the difference $u-v$ is a subharmonic function, which goes to zero at infinity and is equal to zero on $\partial\Omega$. From the weak maximum principle we can conclude

$$u - v \ \leq \ 0,$$

from which (3.3).

If u is a solution, we can take a ball B centered at the origin and a constant C, such that, the subharmonic function $u - C\,\Gamma$ goes to zero at infinity, equals to zero on $\partial\Omega$, and is positive on the (compact) boundary ∂B. From the weak maximal principle $u - C\,\Gamma$ is positive everywhere. The proof of the Theorem is complete. $\square$

Remark 3.2. *The second part of of Theorem 3.1, i.e., when we assume in addition $R\geq 0$, can be derived also with the help of the Kelvin transform on $\mathbb{R}^n$, which can be seen as follows.*

We know that the Kelvin transform is an isometry between $\mathcal{D}^{1,2}(\Omega)$ and $\mathcal{D}^{1,2}(\Omega^*)$, cf. [E] and [GV1]. A calculation using $\triangle(\mathcal{K}u) \ = \ |z|^{-n-2}(\triangle u)\,(\frac{z}{|z|^2})$ shows that the Kelvin transform

$\mathcal{K}u$ of u is a non-negative weak solution of the inequality

$$-\triangle(\mathcal{K}u)(z) \;\leq\; R(\frac{z}{|z|^2})\,\frac{|x|^s}{|z|^{2s}}(\mathcal{K}u)^{2^*(s)-1}(z) \;\leq\; R(\frac{z}{|z|^2})\,\frac{1}{|x|^s}(\mathcal{K}u)^{2^*(s)-1}(z) \qquad \text{in} \quad \Omega^*.$$

Notice that $R(\frac{z}{|z|^2}) \in L^\infty(\Omega^*)$ when $R \in L^\infty(\Omega)$. Furthermore, if Ω is a neighborhood of the infinity then 0 is a removable singularity of $\mathcal{K}u$, i.e., the above inequality is satisfied on $\Omega^* \cup \{0\}$ and $\mathcal{K}u \in \mathcal{D}^{1,2}(\Omega^* \cup \{0\})$, cf. [E] and [GV1]. In other words, when $0 \leq R \in L^\infty$ the function Ku satisfies the same inequality and conditions as u. Recalling that $u(z) = |z|^{2-n}(\mathcal{K}u)(\frac{z}{|z|^2})$ the claim follows from the regularity we have shown in Section 2.

The following lemma contains a useful fact concerning the value of a certain integral, which was used in the above Theorem.

Lemma 3.3. *Let $m > \frac{n-s}{2}$ and $0 \leq s < k < n$. The following formula holds*

$$\int\limits_{\mathbb{R}^k}\int\limits_{\mathbb{R}^{n-k}} \frac{1}{(1+|x|^2+|y|^2)^m}\,\frac{1}{|x|^s}\,dy\,dx = \frac{\sigma_{n-k}}{2}\frac{\sigma_k}{2}\,B(\frac{n-k}{2}, m-\frac{n-k}{2})\;B(\frac{k-s}{2}, m-\frac{n-s}{2}). \tag{3.7}$$

Proof. With $a^2 = 1+|x|^2$ we have

$$\int_{\mathbb{R}^{n-k}} \frac{1}{(1+|x|^2+|y|^2)^m}\,dy \;=\; \frac{a^{n-k}}{a^{2m}}\int_{\mathbb{R}^{n-k}} \frac{1}{(1+|y|^2)^m}\,dy \tag{3.8}$$

$$= \frac{\sigma_{n-k}}{a^{2m-(n-k)}}\int_0^\infty \frac{r^{n-k-1}}{(1+r^2)^m}\,dr \;=\; \frac{\sigma_{n-k}}{2a^{2m-(n-k)}}\int_0^\infty \frac{t^{\frac{n-k}{2}-1}}{(1+t)^{\frac{n-k}{2}+(m-\frac{n-k}{2})}}\,dt$$

$$= \frac{\sigma_{n-k}}{2a^{2m-(n-k)}}\;B(\frac{n-k}{2}, m-\frac{n-k}{2}).$$

Therefore we have

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^{n-k}} \frac{1}{(1+|x|^2+|y|^2)^m}\,\frac{1}{|x|^s}\,dy\,dx \tag{3.9}$$
$$= \frac{\sigma_{n-k}}{2}\,B(\frac{n-k}{2}, m-\frac{n-k}{2})\;\int_{\mathbb{R}^k} \frac{1}{(1+|x|^2)^{m-\frac{n-k}{2}}}\;\frac{1}{|x|^s}\,dx$$

$$= \frac{\sigma_{n-k}}{2}\frac{\sigma_k}{2}\,B(\frac{n-k}{2}, m-\frac{n-k}{2})\;\int_0^\infty \frac{t^{\frac{k-s}{2}-1}}{(1+t)^{\frac{k-s}{2}+(m-\frac{n-k}{2}-\frac{k-s}{2})}}\,dt$$

$$= \frac{\sigma_{n-k}}{2}\frac{\sigma_k}{2}\,B(\frac{n-k}{2}, m-\frac{n-k}{2})\;B(\frac{k-s}{2}, m-\frac{n-s}{2}).$$

$\square$

Remark 3.4. *For future reference, let us notice that the above proof amounts to using twice, with the appropriate choice of the involved parameters, the formula*

$$\int_{\mathbb{R}^k} \frac{1}{(1+|x|^2)^a}\;\frac{1}{|x|^s}\,dx \;=\; \frac{\sigma_k}{2}\;B(\frac{k-s}{2}, a-\frac{k-s}{2}),$$

which is valid for any $a > 0$, $k > s$ *and* $a > \frac{k-s}{2}$.

We end the section with one more technical lemma, which was used in Theorem 3.1.

Lemma 3.5. *Let* $k \geq 2$, $n \geq 3$ *and* $0 \leq\ s < k\ \leq\ n$ *and* $s < 2$. *There exists a constant* $C > 0$ *such that*

$$I(z)\ =\ \int_{|z-\zeta|\leq\frac{|z|}{2}}\ \frac{1}{|z-\zeta|^{n-2}}\frac{1}{|\xi|^s}\,d\zeta\ \leq\ C\ |z|^{2-s}.$$

Proof. To estimate the integral I we observe that I is homogeneous, $I(\lambda z) = \lambda^{2-s}I_1(z)$ for $\lambda > 0$. Therefore, if I is finite on $|z| = 1$ we can conclude that

$$I(z)\ \leq\ C|z|^{2-s}. \tag{3.10}$$

In order to see that I is finite when $|z| = 1$ let us notice that it depends only on $|x|$ and $|y|$ as the integral is invariant under rotation in $\mathbb{R}^k$ or $\mathbb{R}^{n-k}$. A consequence of this fact is that it is enough to show that on $|z| = 1$ the integral I is finite at only three point, namely, $x = 0$, $x = 1$, and $x = y$ (we write $x = 1$ for the number one on the real axis considered as a point in $\mathbb{R}^k$, etc.).

For the rest of the proof we assume $|z| = 1$.

The case of $x \neq 0$ is easier, so we shall consider the last two points first. Without any loss of generality we take $x = 1$ and we split the integral in two parts

$$I(z)\ =\ \int_{\{|z-\zeta|\leq\frac{|z|}{2}\}\cap\{|x-\xi|\leq\frac{|z|}{4}\}}\qquad +\ \int_{\{|z-\zeta|\leq\frac{|z|}{2}\}\cap\{|x-\xi|\geq\frac{|z|}{4}\}}.$$

On the domain of integration of the first integral we have that $|\xi|$ is bounded away from zero and so the integral is finite. In turn, on the domain of integration of the second integral $|z-\zeta|$ is bounded away from zero and $|\xi|^{-s}$ is integrable near the origin as $k > s$, hence this integral is finite again.

Let us consider now the case $x = 0$. Introducing $\rho\xi_o = \xi$ and $r\eta_o = y - \eta$ we put I in the form

$$I\ =\ \int_{|\xi_0|=1}\int_{|\eta_o|=1}\int_{r^2+\rho^2\leq 1/4}\ \frac{\rho^{k-1}\ r^{n-k-1}}{(\rho^2+r^2)^{\frac{n-2}{2}}}\ \frac{1}{\rho^s}\,dr\,d\rho\,d\eta_o\,d\xi_o.$$

Letting $r = t\cos\phi$, $\rho = t\sin\phi$ we come to

$$I\ =\ \sigma_{n-k}\sigma_k\int_0^{1/2} t^{1-s}\,dt\ \int_0^{2\pi}\frac{(\sin\phi)^{k-1}(\cos\phi)^{n-k-1}}{|\sin\phi|^s}\,d\phi\ <\infty,$$

iff $s < 2$ and $k > s$. □

4. A non-linear equation in $\mathbb{R}^n$ related to the Yamabe equation on groups of Heisenberg type

Suppose a and b are two natural numbers, $\lambda > 0$, and for $x, y \in \mathbb{R}^+ = (0, +\infty)$, define the function

$$\phi\ =\ \lambda^2\big[\ (x+\alpha)^2 + (y+\beta)^2\ \big],$$

where $\alpha,\ \beta \in \mathbb{R}$.

Proposition 4.1. *The function ϕ satisfies the following equation in the plane*

$$\Delta\phi \;-\; \frac{a+b+2}{2}\,\frac{|\nabla\phi|^2}{\phi} \;+\; \frac{a}{x}\,\phi_x \;+\; \frac{b}{y}\,\phi_y \;=\; \frac{2a\lambda^2\alpha}{x} \;+\; \frac{2b\lambda^2\beta}{y}, \quad xy\neq 0. \tag{4.1}$$

Proof. Set $\xi=\lambda(x+\alpha)$, $\eta=\lambda(y+\beta)$ and define $\tilde\phi(\xi,\eta)=\phi(x,y)$. Then we have

$$\frac{\partial}{\partial x}=\lambda\frac{\partial}{\partial\xi}\quad\text{and}\quad\frac{\partial}{\partial y}=\lambda\frac{\partial}{\partial\eta}.$$

Thus we have,

$$\begin{aligned}\Sigma \;&\overset{def}{=}\; \Delta\phi \;-\; \frac{a+b+2}{2}\,\frac{|\nabla\phi|^2}{\phi} \;+\; \frac{a}{x}\,\phi_x \;+\; \frac{b}{y}\,\phi_y\\ &=\; \lambda\triangle\tilde\phi \;-\; \frac{a+b+2}{2}\lambda^2\,\frac{|\nabla\tilde\phi|^2}{\tilde\phi} \;+\; \frac{a\lambda}{x}\,\tilde\phi_\xi \;+\; \frac{b\lambda}{y}\,\tilde\phi_\eta.\end{aligned} \tag{4.2}$$

Since $\tilde\phi=\xi^2+\eta^2$ we have

$$\begin{aligned}\Sigma \;&=4\lambda^2 \;-\; \frac{n+2}{2}\lambda^2\frac{4(\xi^2+\eta^2)}{\xi^2+\eta^2} \;+\; \frac{a}{x}2\lambda\xi \;+\; \frac{b}{y}2\lambda\eta\\ &=\; -2n\lambda^2 \;+\; \frac{2a\lambda}{x}(\lambda x+\lambda\alpha) \;+\; \frac{2b\lambda}{y}(\lambda y+\lambda\beta)\\ &=\; -\,2n\lambda^2 \;+\; 2a\lambda^2 \;+\; 2b\lambda^2 \;+\; \frac{2a\lambda^2\alpha}{x} \;+\; \frac{2b\lambda^2\beta}{y}.\end{aligned} \tag{4.3}$$

Hence, taking into account $a+b=n$, we proved

$$\Sigma \;=\; \frac{2a\lambda^2\alpha}{x} \;+\; \frac{2b\lambda^2\beta}{y}.$$

□

Noting that $\Delta\phi \;+\; \frac{a}{x}\,\phi_x \;+\; \frac{b}{y}\,\phi_y$ is the laplacian in $\mathbb{R}^n\equiv\mathbb{R}^{a+1}\times\mathbb{R}^{b+1}$, $n=a+b+2$, acting on functions with cylindrical symmetry, i.e., depending on $|\boldsymbol{x}|$ and $|\boldsymbol{y}|$ only, we are lead to the following question.

Question 4.2. *Given two real numbers p_o and q_o find all positive solutions of the equation*

$$\Delta u \;-\; \frac{n}{2}\,\frac{|\nabla u|^2}{u} \;=\; \frac{p_o}{|\boldsymbol{x}|} \;+\; \frac{q_o}{|\boldsymbol{y}|}, \quad (\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^n\equiv\mathbb{R}^{a+1}\times\mathbb{R}^{b+1},$$

which have at most a quadratic growth condition at infinity, $u\;\leq\;C(|\boldsymbol{x}|^2+|\boldsymbol{y}|^2)$.

As usual a simple transformation allows to remove the appearance of the gradient in the above equation. For a function F we have $\Delta F(u)\;=\;F''(u)|\nabla u|^2\;+\;F'(u)\Delta u$ and thus

$$\begin{aligned}\Delta u^\tau \;&=\; \tau(\tau-1)u^{\tau-2}|\nabla u|^2 \;+\; \tau u^{\tau-1}\Delta u\\ &=\; \frac{\tau}{2}u^{\tau-2}(2u\Delta u \;+\; 2(\tau-1)|\nabla u|^2).\end{aligned}$$

Therefore we choose τ such that $2(\tau-1) = -n$, i.e., $\tau = \frac{2-n}{2}$ and then rewrite the equation for u as

$$\begin{aligned} \Delta u^{\frac{2-n}{2}} &= \frac{2-n}{2} u^{\frac{2-n}{2}-2}(2u\Delta u - n|\nabla u|^2) \\ &= -\frac{(n-2)}{2} u^{\frac{2-n}{2}-1}\Big(\frac{p_o}{|\boldsymbol{x}|} + \frac{q_o}{|\boldsymbol{y}|}\Big). \end{aligned}$$

This is the equation which we will study. As a consequence of the above calculations we can write a three parameter family of explicit solutions.

Proposition 4.3. *Let $\lambda > 0$. The function $v(\boldsymbol{x},\boldsymbol{y})$ defined in $\mathbb{R}^n \equiv \mathbb{R}^{a+1}\times\mathbb{R}^{b+1}$ by the formula*

$$v(\boldsymbol{x},\boldsymbol{y}) = \lambda^{2-n}\big((|\boldsymbol{x}|+\alpha)^2 + (|\boldsymbol{y}|+\beta)^2 \big)^{\frac{2-n}{2}}, \qquad (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^n \equiv \mathbb{R}^{a+1}\times\mathbb{R}^{b+1}, \tag{4.4}$$

satisfies the equation

$$\Delta v = -\, v^{\frac{n}{n-2}}\Big(\frac{p}{|\boldsymbol{x}|} + \frac{q}{|\boldsymbol{y}|}\Big), \tag{4.5}$$

where

$$p = \alpha\,(n-2)\lambda^2\, a, \qquad q = \beta\,(n-2)\lambda^2\, b. \tag{4.6}$$

Let us observe that the above equation is invariant under rotations in the $\boldsymbol{x}$ or $\boldsymbol{y}$ variables. Also if v is a solution then a simple calculations shows that for any $t \neq 0$ the function $v_t(\boldsymbol{x},\boldsymbol{y}) = t^{(n-2)/2}\, v(t\boldsymbol{x}, t\boldsymbol{y})$ is also a solution.

Another observation is that the same principle works if we split $\mathbb{R}^n$ in more than two subspaces. For example, if we take three subspaces we can consider the equation

$$\Delta v = v^{\frac{n}{n-2}} f(|\boldsymbol{x}|,|\boldsymbol{y}|,|\boldsymbol{z}|), \qquad f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \frac{p}{|\boldsymbol{x}|} + \frac{q}{|\boldsymbol{y}|} + \frac{r}{|\boldsymbol{z}|}$$

and ask the question of finding all positive solutions with the same behavior at infinity as the fundamental solution. Clearly the function

$$v = \lambda^{2-n}\big((|\boldsymbol{x}|+\alpha)^2 + (|\boldsymbol{y}|+\beta)^2 + (|\boldsymbol{z}|+\gamma)^2 \big)^{\frac{2-n}{2}},$$

with the obvious choice of α, β and γ is a solution.

5. **The best constant and extremals of the Hardy-Sobolev inequality**

In this Section we give the proof of Theorem 1.2. It was proven in [SSW] that there are extremals with cylindrical symmetry, i.e., functions depending only on $|x|$ and $|y|$ for which the inequality becomes equality. On the other hand, it was shown in [MS] that all extremals of inequality (1.5) have cylindrical symmetry after a suitable translation in the y variable, see also [CW] and [LW] for some related results.

Theorem 5.1 ([MS]). *If $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ is a function for which equality holds in (1.5) then*

i) for any $y \in \mathbb{R}^{n-k}$ the function $u(.,y)$ is a radially symmetric decreasing function in $\mathbb{R}^k$;
ii) there exists a $y_o \in \mathbb{R}^{n-k}$ such that for all $x \in \mathbb{R}^k$ the function $u(x,.+y_o)$ is a radially symmetric decreasing function on $\mathbb{R}^{n-k}$.

We turn to the proof of Theorem 1.2, in which we find the extremals and the best constant in (1.5) in the case $\sigma p_\sigma = 1$, i.e., $s = 1$ in Theorem 1.1.

Proof. (of Theorem 1.2) By Theorem 2.1 and Theorem 2.5 of [BT] there is a constant K for which (1.6) holds and this constant is achieved, i.e., the equality is achieved. A small argument shows that a non-negative extremal u of the naturally associated variational problem $\inf \int_{\mathbb{R}^n} |\nabla u|^2 \, dz$ subject to the constraint

$$\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{|u|^{\frac{2(n-1)}{n-2}}}{|x|} \, dxdy \;=\; 1 \tag{5.1}$$

satisfies the Euler-Lagrange equation

$$\triangle u \;=\; -\frac{\Lambda}{|x|}\, u^{\frac{n}{n-2}}, \qquad u \in D^{1,2}\,(\mathbb{R}^n), \tag{5.2}$$

where $\Lambda \;=\; K^{\frac{2(n-1)}{n-2}}$. From Theorem 2.4 and standard elliptic regularity results we can see that v is a C^∞ function on $|x| \neq 0$. Furthermore, $\nabla u \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and u is C^∞ smooth in the y variables. In particular $u \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$ for any $0 < \alpha < 1$. In order to see these claims let $v = u_{y_j}$ for some j. Hardy's inequality shows $\frac{v}{|x|} \in L^q_{\text{loc}}$ for any $1 < q < k$. From elliptic regularity $v \in W^{2,q}_{\text{loc}}(\mathbb{R}^n)$ for any $1 < q < k$ and hence the Sobolev embedding gives $v \in W^{1,\,\delta q}_{\text{loc}}(\mathbb{R}^n)$, where $\delta = \frac{n}{n-q} > 1$. After finitely many iterations we see that $v \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ from which we can invoke Remark 2.3 to conclude $v \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. The same argument can be done for the higher order derivatives in the y variables. For the x derivatives we argue similarly. We consider ($v = u_{x_i}$)

$$\triangle v \;=\; -\frac{V}{|x|} v \;-\; \frac{V_o}{|x|}.$$

We note that $V,\ V_o \;\in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and it is not hard to see that the proof of Theorem 2.4, cf. also Remarks 2.2 and Remark 2.3, allows to conclude $v \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, and hence $\nabla_x u \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ as claimed.

From Theorem 1.1 of [MS] we can assume that u is an extremal with a cylindrical function after a suitable translation in the y variable. Thus we can assume that u has cylindrical symmetry. Introducing $\rho = |x|$, $r = |y|$ we have that u is a function of ρ and r. We define $U(\rho, r) = u$ by restricting u to two lines through the origin-one in $\mathbb{R}^k$, the other in $\mathbb{R}^{n-k}$. From the regularity of u it follows that U is a smooth function of r for any fixed ρ. For any fixed r it is a smooth function of ρ when $\rho \neq 0$, and Lipschitz for any ρ. Furthermore, in the first quadrant $\rho > 0, r > 0$ of the $\rho\, r$-plane it satisfies the equation

$$\Delta U \;=\; -\,\frac{\Lambda}{\rho}\; U^{\frac{n}{n-2}}\;. \tag{5.3}$$

Using the equation and the smoothness of U in r it is not hard to see that U has bounded first and second order derivatives on ($(0,1) \times (0,1)$), cf. Lemma 5.2.

Let $\phi(\rho, r) \;=\; U^{-\frac{2}{n-2}}$. The calculations of Section 4 show that ϕ satisfies the following equation in the plane

$$\Delta\phi \;-\; \frac{n}{2}\,\frac{|\nabla\phi|^2}{\phi} \;+\; \frac{a}{\rho}\,\phi_\rho \;+\; \frac{b}{r}\,\phi_r \;-\; \frac{2\Lambda}{n-2}\frac{1}{\rho} \;=\; 0, \tag{5.4}$$

where $a = k - 1$, $b = n - k - 1$. Let $\mu > 0$ and consider $\tilde\phi \;=\; \mu^{-1}\phi$. Clearly $\tilde\phi$ is a solution of

$$\Delta\tilde\phi \;-\; \frac{n}{2}\,\frac{|\nabla\tilde\phi|^2}{\tilde\phi} \;+\; \frac{a}{\rho}\,\tilde\phi_\rho \;+\; \frac{b}{r}\,\tilde\phi_r \;-\; \frac{2\Lambda}{\mu(n-2)}\frac{1}{\rho} \;=\; 0.$$

Let us choose μ such that $\frac{2\Lambda}{\mu(n-2)} = \frac{n-2}{2}$, i.e.,

$$\mu = \frac{4\Lambda}{(n-2)^2}.$$

With this choice of μ we see that $\tilde{\phi}$ satisfies equation (4.11) in [GV2]. Moreover, a small argument using the homogeneity of the Kelvin transform shows it satisfies the asymptotic behavior (4.37) of [GV2], except the inequality for the derivatives hold only on $|x| \neq 0$. We can apply (4.40) of [GV2] by noticing that the integrals on the ρ and r axis vanish as U has bounded first and second order derivatives in the punctured neighborhood of any point from the closed first quadrant, a fact which we observed above. Hence (4.43) of [GV2] after setting $|A| = \lambda$ gives

$$\tilde{\phi} = \lambda^2 \big[\, (r + \frac{n-2}{4a\lambda^2})^2 + s^2 \, \big],$$

Recalling that $\phi = \mu \tilde{\phi}$ and the value of μ we come to

$$\phi = \lambda^2 \frac{4\Lambda}{(n-2)^2} \big[\, (r + \frac{n-2}{4a\lambda^2})^2 + s^2 \, \big].$$

This shows that v must equal

$$\begin{aligned} v &= \lambda^{-(n-2)} \Big(\frac{4}{(n-2)^2} \Big)^{-\frac{n-2}{2}} \Lambda^{-\frac{n-2}{2}} \Big[\, (|x| + \frac{n-2}{4a\lambda^2})^2 + |y|^2 \, \Big]^{-\frac{n-2}{2}} \\ &= \lambda^{-(n-2)} \Big(\frac{n-2}{2} \Big)^{n-2} K^{-(n-1)} \Big[\, (|x| + \frac{n-2}{4a\lambda^2})^2 + |y|^2 \, \Big]^{-\frac{n-2}{2}}. \end{aligned}$$

The value of K is determined by (5.1) after fixing λ arbitrarily, say $\lambda = 1$, since the value of the integral in (5.1) is independent of λ. With this goal in mind we set $p = \frac{n-2}{4a}$ and compute the integral

$$\begin{aligned} 1 &= \int_{\mathbb{R}^{n-k}\mathbb{R}^k} \frac{1}{|x|} \Big[\, \Big(\frac{n-2}{2} \Big)^{n-2} \frac{1}{K^{n-1}} \frac{1}{\big[\, (|x|+p)^2 + |y|^2 \, \big]^{\frac{n-2}{2}}} \, \Big]^{\frac{2(n-1)}{n-2}} \, dxdy \\ &= \frac{1}{K^{\frac{2(n-1)^2}{n-2}}} \Big(\frac{n-2}{2} \Big)^{2(n-1)} \int_{\mathbb{R}^{n-k}\mathbb{R}^k} \frac{1}{|x|} \frac{1}{\big[\, (|x|+p)^2 + |y|^2 \, \big]^{n-1}} \, dxdy \end{aligned} \tag{5.5}$$

Let $a = |x| + p$. Then we compute

$$\begin{aligned} \int_{\mathbb{R}^{n-k}} \frac{1}{(a^2 + |y|^2)^{n-1}} \, dy &= \frac{1}{a^{n+k-2}} \int_{\mathbb{R}^{n-k}} \frac{1}{(1 + |y|^2)^{n-1}} \, dy \\ &= \frac{\sigma_{n-k}}{2a^{n+k-2}} \, B(\frac{n-k}{2}, \frac{n+k}{2} - 1), \end{aligned} \tag{5.6}$$

where σ_{n-k} is the volume of the unit $n-k$ dimensional sphere and $B(.,.)$ is the beta function. On the other hand after a simple computation we find

$$\begin{aligned} \int_{\mathbb{R}^k} \frac{1}{|x|(|x|+p)^{n+k-2}} \, dx &= \frac{\sigma_k}{p^{n+k+1}} \int_0^\infty \frac{r^{k-2}}{(r+1)^{n+k-2}} \, dr \\ &= \frac{\sigma_k}{p^{n+k+1}} B(k-1, n-1). \end{aligned} \tag{5.7}$$

Plugging in (5.5) come to

$$
\begin{aligned}
K^{\frac{2(n-1)^2}{n-2}} &= \Big(\frac{n-2}{2}\Big)^{2(n-1)} \frac{\sigma_{n-k}}{2}\ B(\frac{n-k}{2}, \frac{n+k}{2}-1)\frac{\sigma_k}{p^{n+k+1}}\ B(k-1, n+k-1) \\
&= 2^{2k+3}(n-2)^{n-k-3}(k-1)^{n+k+1}\sigma_{n-k}\sigma_k B(\frac{n-k}{2}, \frac{n+k}{2}-1)B(k-1, n-1).
\end{aligned} \tag{5.8}
$$

The proof is complete taking into account the allowed translations in the y variable. $\square$

In the above proof we used the following simple ODE lemma, which can be proved by integrating the equation.

Lemma 5.2. *Suppose f is a smooth function on $\mathbb{R}\setminus\{0\}$, which is also locally Lipschitz on $\mathbb{R}$, i.e., on any compact interval there is a constant L, such that, $|f(t') - f(t'')| \leq\ L\,|t' - t''|$ for any two points t', t'' on this interval. If f satisfies the equation*

$$
f'' + \frac{k}{t}f' \ =\ \frac{a}{t}\ +\ b, \qquad t > 0,
$$

where k is a constant $k > 1$ and a, b are L^∞_{loc} functions, then f has bounded first and second order derivatives near the origin.

6. **Some applications**

Let us consider the prescribed scalar curvature equation on $\mathbb{R}^n$

$$
\triangle u \ =\ -R(z)\ u^{2^*-1}, \tag{6.1}
$$

where R is a bounded function and u is a non-negative function. As usual we say that u is of finite energy if $\|u\|_{\mathcal{D}}{}^{1,p}(\mathbb{R}^n)$ is finite. Clearly if $g = u^{4/(n-2)}g_o$ is a metric conformal to the Euclidean metric g_o on $\mathbb{R}^n$ then the finite energy condition is equivalent to g having finite volume. The results of Section 2 can be extended to the case of the scalar curvature equation of many non-compact manifold with positive Yamabe invariant, which among other things will be done in [VZ], but the following result, which follows from the fast decay Theorem 3.1 (a) of u, i.e., at least as fast as the fundamental solution, is indicative of what is to be expected, see also [Le].

Theorem 6.1. *Suppose $R \in L^\infty$ and g_o is the Euclidean metric on $\mathbb{R}^n$. Let u be a positive solution to* (6.1)*. If $\mathbb{R}^n$ with the conformal metric $g = u^{4/(n-2)}g_o$ has finite volume then u has fast decay and the metric g is incomplete.*

The second application concerns the original motivation of Badiale and Tarantello [BT] to consider the Hardy-Sobolev inequality. The following equation has been proposed, cf. [Ch], [B] and [R] for further details, as a model to study elliptic galaxies.

$$
-\triangle u \ =\ \phi(|x|)u^{q-1}, \quad 0\ <\ u \in D^{1,2}(\mathbb{R}^3) \quad (\ z = (x, y) \in \mathbb{R}^3\ !). \tag{6.2}
$$

It is also required that u is of finite mass, i.e.,

$$
\int \phi u^{q-1}\, dz\ <\ \infty.
$$

Using the results of this paper we can show the following Theorem.

Theorem 6.2. *Suppose $(1+|x|)^\gamma \phi \in L^\infty(\mathbb{R}^3)$ for some $0<\gamma<2$. If $2^*(\gamma)<q<6$, then any solution of* (6.2) *is also of finite mass.*

Proof. Since the dimension of the ambient space is three we have $2^*(0)=6$ and $2<2^*(\gamma)<6$. Given any q satisfying $2^*(\gamma)<q<6$ we can find an $s<\gamma$, such that, $q=2^*(s)$ and u satisfies the equation

$$-\triangle u \;=\; \phi(|x|)u^{q-1} \;=\; V\frac{|u|^{2^*(s)-1}}{|x|^s}$$

with $|V| = |x|^s|\phi| \leq (1+|x|)^s|\phi| \;\leq (1+|x|)^\gamma|\phi| \in L^\infty(\mathbb{R})$. Theorem 3.1 (a) implies that u decays at least as fast as the fundamental solution of the laplacian in $\mathbb{R}^3$

$$u(z) \;\leq\; \frac{C}{1+|z|}.$$

Since $q-1>2^*(\gamma)-1>1$ it follows

$$\int \phi u^{q-1}\,dz \;<\; \infty,$$

which shows that every finite energy solution is also of finite mass. □

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